Physics 216,Quiz 2

November 29, 2016, 90 minutes

Do 8 problems out of 10.

1. A function $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$. Is this function even or odd? Find the Fourier series.

2. The Fourier series of the function $f(x)$ of period $L = 2$ which is equal to x on the interval $(-1, 1)$ is given by

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}\sin n\pi x$$

   Use this expansion and Parseval theorem to show that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

3. Find the Fourier transform $F(\omega)$ of the function $f(t) = e^{-a|t|}$ where $a > 0$.

4. A rectangular pulse is represented by $f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$. Find Fourier transform and use the convolution theorem $\int_{-\infty}^{\infty} f(t) f^*(t)\, dt = \int_{-\infty}^{\infty} F(\omega) F^*(\omega)\, d\omega$ to evaluate the integral $\int_0^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$.

5. Consider the Sturm-Liouville operator $\pounds y = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$. Find the weight function $w(x)$ for this operator to be self-adjoint.

6. Use the recursion relation of the Legendre polynomials

$$(2n+1)\, xP_n(x) = (n+1)\, P_{n+1}(x) + nP_{n-1}(x)$$

   and the values $P_0 = 1$, $P_1 = x$ to prove by induction that $P_n(-x) = (-1)^n P_n(x)$.

7. Given a set of orthogonal functions $\{Q_n(x)\}$ satisfying the Sturm-Liouville equation

$$\pounds Q_n = \frac{1}{w}\frac{d}{dx}\left(w\frac{dQ_n}{dx}\right) + \gamma Q_n = 0 \qquad (1)$$

   with weight functions $w(x)$ so that $\int w(x)\, Q_n^*(x)\, Q_m(x)\, dx = 0$, if $n \neq m$. Show that the functions $\{Q_n'(x)\}$ are orthogonal functions with weight $w(x)\,\alpha(x)$. Hint: Substitute in the integral $\int w(x)\, Q_n^*(x)\, \pounds Q_m(x)\, dx = 0$.

8. Express the function $f(\theta,\phi) = \sin\theta\left(\sin^2\frac{\theta}{2}\cos\phi + i\cos^2\frac{\theta}{2}\sin\phi\right) + \sin^2\frac{\theta}{2}$ as a sum of spherical harmonics. Use the values of $Y_l^m(\theta,\phi)$ given in Table 15.4 page 760 in Arfken.

9. Hermite polynomials are defined by $g(x,t) = \exp\left(-t^2 + 2tx\right) = \sum_{n=0}^{\infty} t^n H_n(x)$. Use this to prove that $H_n(-x) = (-1)^n H_n(x)$. Multiply $g(x,t)$ by $e^{-\frac{x^2}{2}}$ and integrate to show that $\int_0^\infty e^{-\frac{x^2}{2}} H_n(x)\, dx = 0$ if $n = 2m+1$.

10. Use the relation

$$xL_n^k = (2n+k+1) L_n^k - (n+k) L_{n-1}^k - (n+1) L_{n+1}^k$$

to show that

$$\int_0^\infty e^{-x} x^{k+1} L_n^k(x) L_n^k(x)\, dx = \frac{(n+k)!}{n!} (2n+k+1).$$